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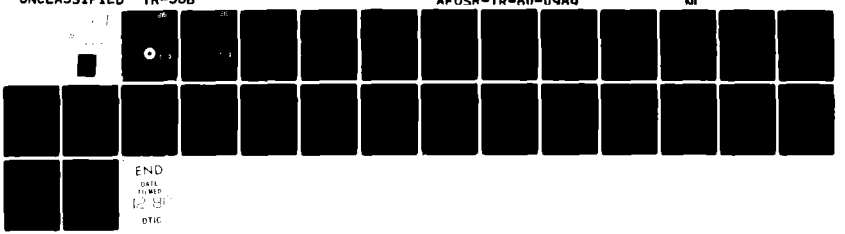

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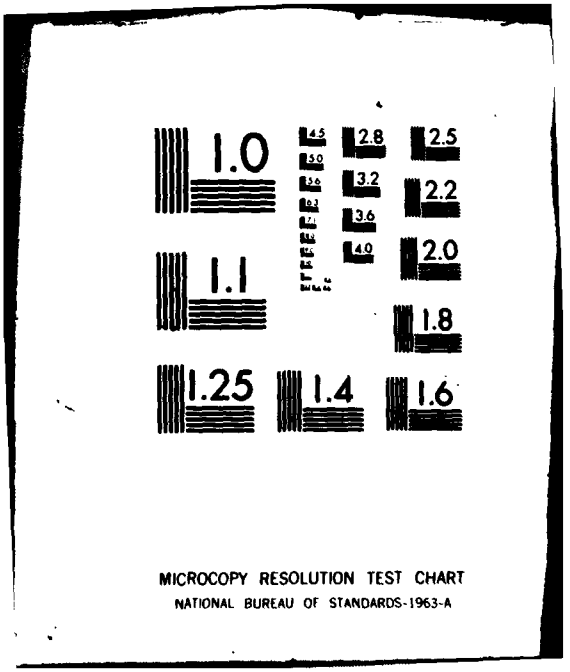
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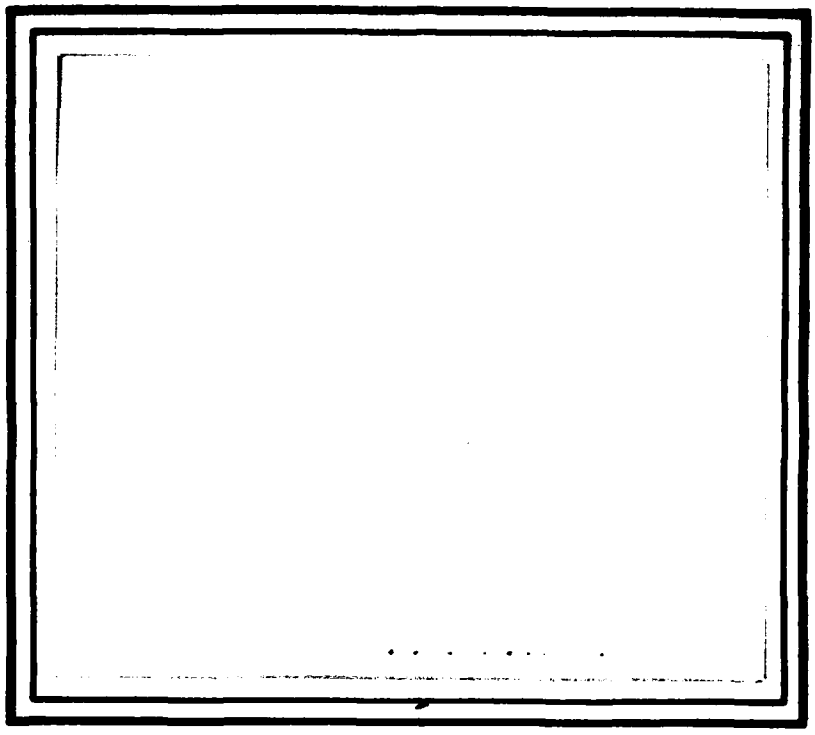

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A PHASE TYPE, SEMI-MARKOVIAN POINT PROCESS

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ABSTRACT

A semi-Markovian point process is defined, for which the intervals of time between successive events have phase type distribution. The distribution of the number of events in an interval is examined, and it is shown how the expected number of events in an interval may be efficiently computed. A stationary version of the process is analysed. In particular, the necessary and sufficient condition under which the new process is a renewal process is determined. Simple sufficient conditions are presented.

Keywords

Semi-Markov process, phase type distributions, conditions for independence, computational probability.

Introduction

Most of the literature on queueing theory deals with systems for which arrivals occur according to a renewal process. From the available results, it is clear that the analysis of a queueing system with a general non-renewal arrival process is very difficult.

In the present paper, we define and discuss a special semi-Markovian point process. It is assumed that there are N different types of intervals, each with a phase type distribution. The types of successive intervals are determined by a Markov chain with transition probability matrix P. If a given interval is of type i, $1 \leq i \leq N$, then the following interval is of type j, $1 \leq j \leq N$, with probability P_{ij}. Phase type distributions have great versatility and the structure of the process is very simple. It should, therefore, be a useful tool in modeling queueing systems with non-independent arrivals, and provide analytically or algorithmically tractable results.

We shall give a formal definition of the process in the next section. We examine in Section 2 the number of events in an interval and show how the expected number of events may be efficiently computed. In Section 3, we analyse the correlation structure of a stationary version of the process. In particular, we determine the necessary and sufficient condition under which that process is a renewal process. In Section 4, some examples such as the interrupted Poisson arrivals are considered in further detail.

Notational convention. All vectors are represented by underlined letters. The context indicates whether they are row or column vectors. In order to facilitate the reading of the formulas, the expression $\underline{v}\,\underline{w}$ always represents the inner product of a row vector $\underline{v}$ by a column vector $\underline{w}$; the expression $\underline{w} \cdot \underline{v}$ always represents the product of a column vector

$\underline{w}$ by a row vector $\underline{v}$, yielding a matrix whose (i,j)th element is $w_i v_j$.

1. The Phase Type Semi-Markovian Point Process

1.1 Phase Type Distributions

Phase type distributions have been introduced by Neuts [4]. Consider a (n+1)-state continuous-parameter Markov process, with n transient states and one absorbing state. Its infinitesimal generator Q is of the form

$$Q = \begin{pmatrix} T & \underline{T}^0 \\ \underline{0} & 0 \end{pmatrix},$$

where T is a square matrix of order n, with $T_{ii} < 0$, $T_{ij} \geq 0$, for $i \neq j$, and such that T^{-1} exists. The n-vector $\underline{T}^0$ has nonnegative entries, and is equal to $-T\underline{e}$. The vector $\underline{e}$ has all entries equal to one. The vector of initial probabilities is denoted by $(\underline{\alpha},\alpha_{n+1})$, and satisfies $\underline{\alpha}\,\underline{e} + \alpha_{n+1} = 1$, $0 \leq \alpha_{n+1} < 1$.

The probability distribution $F(\cdot)$ of the time till absorption in the state $n+1$ is then given by

$$F(x) = 1 - \underline{\alpha} \exp(Tx)\, \underline{e}\,, \qquad \text{for } x \geq 0\,.$$

The probability distribution $F(\cdot)$ is said to be of phase type (in short, "F is PH"). The pair $(\underline{\alpha},T)$ is called a representation of $F(\cdot)$. In this paper, we assume that $\alpha_{n+1} = 0$, so that $F(\cdot)$ does not have a jump at 0. Furthermore, we assume that the representation is such that each state has a positive probability of being visited before absorption. Under that assumption, the Markov chain with generator $T + \underline{T}^0 \cdot \underline{\alpha}$ is irreducible.

The moments $\mu^{(k)}$ of $F(\cdot)$ about the origin all exist and are given by

$$\mu^{(k)} = (-1)^k \, k! \, \underline{\alpha} \, T^{-k} \, \underline{e} \, , \qquad \text{for } k \geq 1 \, . \tag{1}$$

1.2 The Point Process

We consider N PH-distributions, with representations $(\underline{\alpha}_i, T_i)$, where T_i is a square matrix of order n_i, for $i = 1, \ldots, N$, and an N-state irreducible Markov chain with transition matrix P. If the Markov chain has made a transition to the state i, the next transition is to the state j, with probability p_{ij}, and the time between these transitions has a PH-distribution $F_i(\cdot)$, with representation $(\underline{\alpha}_i, T_i)$, independent of j. The epochs of transitions for the Markov chain correspond to the epochs of events for the point process.

We denote respectively by $N(t)$, $C(t)$ and $\Phi(t)$, the number of events in $(0,t]$, the state of the Markov chain P at time t, and the state of the Markov chain $T_{C(t)}$, at time t. In other words, suppose that the last event before t occurred at time τ. At time τ, the Markov chain P made a transition to the state $C(\tau) = j$, say, and an initial state was chosen for the Markov chain T_j, according to the probability vector $\underline{\alpha}_j$. In the interval $(\tau,t]$, the Markov chain T_j underwent zero, one or more than one transitions, without entering its absorbing state. At time t, $C(t) = j$, and the Markov chain T_j is in the state $\Phi(t)$.

We make the following independence assumption. For every $t > 0$, the intervals of time between events are conditionally independent, given the path function of the Markov chain P. It is then clear that the process $\{N(t), C(t), \Phi(t), t \geq 0\}$ is a Markov process with state space

$\{(\nu,j,\phi);\ \nu \geq 0,\ 1 \leq j \leq N,\ 1 \leq \phi \leq n_j\}$. In order to distinguish easily between the Markov chain P and the Markov chains T_i, $i = 1, \ldots, N$, we shall refer to the states of any Markov chain T_i as "phases".

2. The Number of Events in an Interval

We define the probabilities $S_{i,\xi;j,\phi} = P[N(t) = \nu,\ C(t) = j,\ \Phi(t) = \phi \mid C(0) = i,\ \Phi(0) = \xi]$, and order the elements $\{(j,\phi);\ 1 \leq j \leq N,\ 1 \leq \phi \leq n_j\}$ as follows: $(1,1), (1,2), \ldots, (1,n_1), (2,1), \ldots, (2,n_2), \ldots, (N,1), \ldots, (N,n_N)$. Finally we define the block-partitioned square matrix $S(\nu,t)$ of order $n_1 + n_2 + \ldots + n_N$ by

$$S(\nu,t) = \begin{pmatrix} S_{1,1}(\nu,t) & S_{1,2}(\nu,t) & \ldots & S_{1,N}(\nu,t) \\ S_{2,1}(\nu,t) & S_{2,2}(\nu,t) & \ldots & S_{2,N}(\nu,t) \\ \vdots & \vdots & & \vdots \\ S_{N,1}(\nu,t) & S_{N,2}(\nu,t) & \ldots & S_{N,N}(\nu,t) \end{pmatrix},$$

where the blocks $S_{i,j}(\nu,t)$ have n_i rows and n_j columns, and the (ξ,ϕ)th element of $S_{i,j}(\nu,t)$ is equal to $S_{i,\xi;j,\phi}(\nu,t)$.

The Chapman-Kolmogorov equations for the process $\{N(t), C(t), \Phi(t), t \geq 0\}$ may be written in matrix notation as

$$\begin{aligned} S'_{i,j}(0,t) &= 0, && \text{for } i \neq j, \\ &= S_{i,i}(0,t)\, T_i, && \text{for } i = j, \\ S'_{i,j}(\nu,t) &= S_{i,j}(\nu,t)\, T_j - \sum_{k=1}^{N} S_{i,k}(\nu-1,t)\, p_{kj}\, \underline{T}^{\circ}_k \cdot \underline{\alpha}_j, \\ & && \text{for } \nu \geq 1. \end{aligned}$$

Therefore, the matrices $S(\nu,t)$ satisfy the system of linear differential

equations:

$$S'(0,t) = S(0,t)\,\overline{T}\,,$$
$$S'(\nu,t) = S(\nu,t)\,\overline{T} - S(\nu-1,t)\,\overline{T}\,\overline{A}\,, \quad \text{for } \nu \geq 1\,,$$

with initial conditions $S(0,0) = I$, $S(\nu,0) = 0$, for $\nu \geq 1$, where the square matrices $\overline{T}$ and $\overline{A}$ are of order $n_1 + n_2 + \ldots + n_N$. The matrix $\overline{T}$ is block-diagonal and given by

$$\overline{T} = \begin{pmatrix} T_1 & 0 & 0 & \ldots & 0 \\ 0 & T_2 & 0 & \ldots & 0 \\ 0 & 0 & T_3 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \\ 0 & 0 & 0 & & T_N \end{pmatrix},$$

and the block-partitioned matrix $\overline{A}$ is given by

$$\overline{A} = \begin{pmatrix} p_{11}\underline{e}_1\cdot\underline{\alpha}_1 & p_{12}\underline{e}_1\cdot\underline{\alpha}_2 & \cdots & p_{1N}\underline{e}_1\cdot\underline{\alpha}_N \\ p_{21}\underline{e}_2\cdot\underline{\alpha}_1 & p_{22}\underline{e}_2\cdot\underline{\alpha}_2 & \cdots & p_{2N}\underline{e}_2\cdot\underline{\alpha}_N \\ \vdots & \vdots & & \vdots \\ p_{N1}\underline{e}_N\cdot\underline{\alpha}_1 & p_{N2}\underline{e}_N\cdot\underline{\alpha}_2 & \cdots & p_{NN}\underline{e}_N\cdot\underline{\alpha}_N \end{pmatrix}.$$

By $\underline{e}_i$ we denote an n_i-vector with each entry equal to one.

The matrix-generating function $\tilde{S}(z,t) = \sum_{\nu=0}^{\infty} z^{\nu} S(\nu,t)$, defined for $|z| \leq 1$, satisfies the differential equation

$$\frac{\partial}{\partial t}\tilde{S}(z,t) = \tilde{S}(z,t)\,\overline{T}\,(I-z\overline{A})\,, \quad \tilde{S}(z,0) = I, \quad \text{for } t \geq 0\,.$$

Hence, we have that $\tilde{S}(z,t) = \exp[\overline{T}(I-z\overline{A})t]$. In particular, $\tilde{S}(0,t) = \exp(\overline{T}t)$, as is to be expected. Also, $\tilde{S}(1,t) = \exp[\overline{T}(I-\overline{A})t]$, which is

again obvious, since the process $\{C(t), \Phi(t), t \geq 0\}$ is a continuous parameter Markov chain with infinitesimal generator $\overline{T}(I-\overline{A})$.

We now define the matrix $M(t) = \left[\frac{\partial}{\partial z} \tilde{S}(z,t)\right]_{t=1}$, and the vector $\underline{m}(t) = M(t)\underline{e}$. We partition that vector as $\underline{m}(t) = (\underline{m}_1(t), \underline{m}_2(t), \ldots, \underline{m}_N(t))$, where $\underline{m}_i(t)$, $i = 1,\ldots,N$, has n_i components. The component $m_{i,\xi}(t)$ is the expected number of events occurring before time t, given the initial conditions $C(0) = i$, and $\Phi(0) = \xi$. Furthermore, we denote by $\underline{\gamma}$ the invariant probability vector associated with P, i.e. $\underline{\gamma} P = \underline{\gamma}$, $\underline{\gamma}\,\underline{e} = 1$. Every entry of $\underline{\gamma}$ is strictly positive. Finally, we denote by $\underline{\pi}$ the stationary probability vector of $\overline{T}(I-\overline{A})$, i.e. $\underline{\pi}\,\overline{T}(I-\overline{A}) = \underline{0}$, $\underline{\pi}\,\underline{e} = 1$.

Lemma 1.

The vector $\underline{m}(t)$ is given by

$$\underline{m}(t) = m^* t\,\underline{e} - (I-\Pi)[\tau^*\Pi - \overline{T}(I-\overline{A})]^{-1}\,\overline{T}\,\underline{e} - [\Pi - \exp(\overline{T}(I-\overline{A})t)]\,[\tau^*\Pi - \overline{T}(I-\overline{A})]^{-1}\,\overline{T}\,\underline{e}\,, \quad \text{for } t \geq 0\,, \tag{2}$$

where the square matrix Π of order $n_1 + n_2 + \ldots + n_N$ is equal to $\underline{e} \cdot \underline{\pi}$. τ^* is any real number such that $\tau^* \geq \max\{-(\overline{T}(I-\overline{A}))_{i,\phi;i,\phi};\ 1 \leq i \leq N,\ 1 \leq \phi \leq n_i\}$, and m^* is given by

$$m^* = -\underline{\pi}\,\overline{T}\,\underline{e}\,. \tag{3}$$

Moreover, if we partition $\underline{\pi}$ as $\underline{\pi} = (\underline{\pi}_1, \underline{\pi}_2, \ldots, \underline{\pi}_N)$, where $\underline{\pi}_i$ is an n_i-vector, $i = 1, \ldots, N$, we have that

$$\underline{\pi}_i = c\,\gamma_i\,\underline{\alpha}_i\,(-T_i)^{-1}\,, \quad \text{for } i = 1, \ldots, N\,. \tag{4}$$

The normalizing constant c satisfies

$$m^* = c = (\underline{\gamma}\,\underline{\mu}^{(1)})^{-1}\,, \tag{5}$$

where the N-vector $\underline{\mu}^{(1)}$ has components $\mu_i^{(1)} = -\underline{\alpha}_i T_i^{-1} \underline{e}$.

Proof. To prove the first part of the lemma, we observe that the point process under consideration is a special case of the "Versatile Markovian Point Process" defined in Neuts [5]. Equation (2) then follows by adapting Equation (12) of [5] to our process. The proof of the second part is immediate.

Remarks

1. The third term in (2) tends to zero as t tends to infinity, since $\Pi - \exp[\overline{T}(I-\overline{A})t] = \Pi - \exp[\hat{S}(1,t)]$, does. Therefore the first two terms give the linear asymptote of $\underline{m}(t)$.

2. In order to compute $\underline{m}(t)$, it is not necessary to evaluate the inverse of the large matrix $[\tau^*\Pi - \overline{T}(I-\overline{A})]$. It suffices to determine the vector $\underline{u}$ defined as

$$\underline{u} = -[\tau^*\Pi - \overline{T}(I-\overline{A})]^{-1} \overline{T}\,\underline{e} . \qquad (6)$$

This may be done efficiently as we show below.

3. The main problem in computing $\underline{m}(t)$ from (2) therefore lies in evaluating the third term, which we denote by $\underline{v}(t)$. The vector $\underline{v}(t)$ is the solution of the system of differential equations, of order $n_1 + n_2 + \ldots + n_N$, given by

$$\underline{v}'(t) = \overline{T}(I-\overline{A})\,\underline{v}(t) , \quad \underline{v}(0) = (\Pi - I)\,\underline{u} .$$

Lemma 2

If we partition the vector $\underline{u}$ as $\underline{u} = (\underline{u}_1, \underline{u}_2, \ldots, \underline{u}_N)$, where $\underline{u}_i$ is an n_i-vector for $i = 1, \ldots, N$, then

$$\underline{u}_i = \hat{\nu}\,\underline{e} + \nu_i\,\underline{e} + \underline{m}^* T_i^{-1} \underline{e} , \qquad \text{for } i = 1, \ldots, N , \qquad (7)$$

where $\underline{\nu} = -m^* (I-P+\Gamma)^{-1} P \underline{\mu}^{(1)}$,

$$\tilde{\nu} = m^*/\tau^* + \frac{1}{2} m^{*2} \underline{\gamma} \underline{\mu}^{(2)} - \underline{h} \underline{\nu} .$$

The vectors $\underline{\mu}^{(2)}$ and $\underline{h}$ have N components, given by $\mu_i^{(2)} = 2 \underline{\alpha}_i T_i^{-2} \underline{e}$, $h_i = m^* \gamma_i \mu_i^{(1)}$, for $i = 1, \ldots, N$.

Proof. The vector $\underline{u}$ is the unique solution to the system

$$[\tau^*\Pi - \bar{T}(I-\bar{A})] \underline{u} = - \bar{T} \underline{e} . \tag{8}$$

Upon substitution of the stated expressions for $\underline{u}$ and after some routine, but belabored calculations, it is verified that (7) indeed provides the solution to (8). The details are omitted for the sake of brevity.

The equation (2) now becomes

$$\underline{m}(t) = m^* t \underline{e} + m^*(I-\Pi)(\bar{T}^{-1}-N)\underline{e} + m^*[\Pi-\exp(\bar{T}(I-\bar{A})t)](\bar{T}^{-1}-N)\underline{e} , \quad \text{for } t \geq 0 , \tag{9}$$

where the matrix N is block-diagonal and given by

$$N = \begin{pmatrix} N_1 & 0 & \cdots & 0 \\ 0 & N_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & N_N \end{pmatrix} ,$$

N_i is a square matrix of order n_i, and is equal to $[(I-P+\Gamma)^{-1} P \underline{\mu}^{(1)}]_i I$.

Remark.

In order to determine $(\bar{T}^{-1}-N)\underline{e}$, it is not necessary to invert matrices. One merely solves $N+1$ systems of linear equations, i.e. the systems $T_i \underline{x}_i = \underline{e}$, for $i = 1, \ldots, N$, and the system $(I-P+\Gamma)\underline{y} = P \underline{\mu}^{(1)}$.

3. A Stationary Version of the Point Process

Usually, the stationary point process, which we denote by P^*, is obtained by choosing the initial state $(C(0),\Phi(0))$ according to the probability vector $\underline{\pi}$. We consider a slightly different process, denoted by $\tilde{P}$, for which $(C(0),\Phi(0))$ is chosen by $P[C(0) = j,\Phi(0) = \xi] = \gamma_j(\underline{\alpha}_j)_\xi$. In other words, we choose the time origin so that at time 0-, an event has occured, the type of the next interval is chosen according to the stationary vector $\underline{\gamma}$ of P. In view of our ultimate objective of using this process to model arrivals to queueing systems, the process $\tilde{P}$ has the following interesting property.

Let us denote by X_n the interval of time between the (n-1)st and the nth event (between time 0 and the first event, if $n = 1$).

Lemma 3

The random variables $\{X_n, n \geq 1\}$ have a common marginal distribution $r(\cdot)$. The distribution $r(\cdot)$ is PH, and has a representation $(\underline{a},\overline{T})$, where the vector $\underline{a}$ has $n_1 + n_2 + \ldots n_N$ components and is given by $\underline{a} = (\gamma_1\underline{\alpha}_1, \gamma_2\underline{\alpha}_2, \ldots, \gamma_N\underline{\alpha}_N)$. Therefore

$$r(x) = 1 - \sum_{i=1}^{N} \gamma_i \, \underline{\alpha}_i \exp(T_i x) \, \underline{e} , \qquad \text{for } x \geq 0 ,$$

and the kth moment $m^{(k)}$ of $r(\cdot)$ about the origin is equal to

$$m^{(k)} = \underline{\gamma} \, \underline{\mu}^{(k)} , \qquad \text{for } k \geq 1 .$$

The N-vectors $\underline{\mu}^{(k)}$ have entries $\mu_i^{(k)} = (-1)^k \, k! \, \underline{\alpha}_i \, T_i^{-k} \, \underline{e}$.

Proof. The proof is obvious.

In the remainder of this section, we examine the correlation structure of the intervals of time between events. This is related to recent work by Simon [7]. Because of the difference in our approach, we postpone discussion

of this relation to the end of this section.

We now introduce the notion of linear dependence for PH-distributions.

Definition 1

The set $\{(\underline{\alpha}_i, T_i),\ 1 \leq i \leq N\}$ is a set of linearly independent PH-distributions if and only if $\sum_{i=1}^{N} d_i F_i(x) = 0$, for all $x \geq 0$, implies that d_i is equal to zero, for $i = 1, \ldots, N$, where $F_i(x) = 1 - \underline{\alpha}_i \exp(T_i x)\underline{e}$, for $x \geq 0$.

Definition 2

The PH-distribution $(\underline{\beta}, B)$ is linearly dependent of the PH-distributions $(\underline{\alpha}_i, T_i)$, $i = 1, \ldots, N$, if and only if there exist $\{d_1, d_2, \ldots, d_N\}$, such that $d_i \neq 0$ for some i, and $1 - \underline{\beta} \exp(Bx)\, \underline{e} = \sum_{i=1}^{N} d_i F_i(x)$, for all $x \geq 0$, where the $F_i(x)$ are defined above.

It is clear that for any such set $\{d_1, \ldots, d_N\}$ we have $\sum_{i=1}^{N} d_i = 1$.

Remarks

1. The term "linearly independent" has been chosen for the following reason. We easily observe that $\sum_{i=1}^{N} d_i F_i(x)$ is equal to zero for all positive x if and only if $\sum_{i=1}^{N} d_i \underline{\alpha}_i T_i^k \underline{e}$ is equal to zero for all $k \geq 1$. The definition 1 is therefore equivalent to the condition that the infinite vectors $(\underline{\alpha}_i T_i \underline{e}, \underline{\alpha}_i T_i^2 \underline{e}, \ldots)$, $i = 1, \ldots, N$, are linearly independent.

2. If $(\underline{\beta}, B)$ is linearly dependent of $\{(\underline{\alpha}_i, T_i),\ i = 1, \ldots, N\}$, and $d_i \geq 0$, for all $i = 1, \ldots, N$, then $(\underline{\beta}, B)$ and $(\underline{\beta}_0, B_0)$ are two representations of the same PH-distribution, where

$$\underline{\beta}_0 = (d_1\underline{\alpha}_1, d_2\underline{\alpha}_2, \ldots, d_N\underline{\alpha}_N),$$

and

$$B_0 = \begin{pmatrix} T_1 & 0 & \cdots & 0 \\ 0 & T_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & T_N \end{pmatrix} .$$

The proof is elementary. Similarly, if $d_i < 0$, for $i = 1, \ldots, J$, and $d_i \geq 0$, for $i = J + 1, \ldots, N$, then clearly

$$1 + \sum_{i=1}^{J} |d_i| = \sum_{i=J+1}^{N} d_i .$$

If we set the latter quantity equal to $\tilde{d}$, then $(\underline{\beta}_1, B_1)$ and $(\underline{\beta}_2, B_2)$ where

$$\underline{\beta}_1 = \left(\frac{1}{\tilde{d}} \underline{\beta}, \frac{|d_1|}{\tilde{d}} \underline{\alpha}_1, \ldots, \frac{|d_J|}{\tilde{d}} \underline{\alpha}_J \right) ,$$

$$B_1 = \begin{pmatrix} B & 0 & \cdots & 0 \\ 0 & T_1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & T_J \end{pmatrix} ,$$

$$\underline{\beta}_2 = \left(\frac{d_{J+1}}{\tilde{d}} \underline{\alpha}_{J+1}, \ldots, \frac{d_N}{\tilde{d}} \underline{\alpha}_N \right) ,$$

$$B_2 = \begin{pmatrix} T_{J+1} & 0 & \cdots & 0 \\ 0 & T_{J+2} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & T_N \end{pmatrix} ,$$

are two representations of the same PH-distribution.

Let now the PH-distributions $\{(\underline{\alpha}_i, T_i),\ i = 1, \ldots, N\}$ which define the point process, be expressed as linear combinations of the linearly independent PH-distributions $\{(\underline{\zeta}_j, Z_j),\ j = 1 \ldots L\}$, with $L \leq N$, i.e.

$$F_i(x) = \sum_{j=1}^{L} d_{ij}\, G_j(x) \quad , \qquad \text{for all } x \geq 0, \text{ and } i = 1, \ldots, N,$$

where $G_j(x) = 1 - \underline{\zeta}_j \exp(Z_j x)\underline{e}$. We denote by D the matrix with (i,j)th element equal to d_{ij}. We also define the vectors $\underline{F}(x) = (F_1(x), F_2(x), \ldots, F_N(x))$, $\underline{G}(x) = (G_1(x), G_2(x), \ldots, G_L(x))$, and $\underline{\mu}^{*(k)} = (\mu_1^{*(k)}, \ldots, \mu_L^{*(k)})$, for $k \geq 1$, where $\mu_j^{*(k)}$ is the kth moment about the origin of $G_j(\cdot)$.

We then clearly have that

$$\underline{F}(x) = D\, \underline{G}(x) \quad , \qquad \text{for } x \geq 0 \quad , \tag{10}$$

and

$$\underline{\mu}^{(k)} = D\, \underline{\mu}^{*(k)} \quad , \qquad \text{for } k \geq 1 \quad . \tag{11}$$

It is also clear that $D\,\underline{e} = \underline{e}$.

We emphasize that the set of PH-distributions $\{(\underline{\zeta}_j, Z_j), j = 1, \ldots, L\}$ is not necessarily a subset of $\{(\underline{\alpha}_i, T_i), i = 1, \ldots, N\}$. The following is an illustrative example.

If

$$(\underline{\alpha}_1, T_1) \quad \left\{(1/2, 1/2),\ \begin{bmatrix} -\lambda_1 & 0 \\ 0 & -\lambda_2 \end{bmatrix}\right\} ,$$

$$(\underline{\alpha}_2, T_2) = \left\{(2/3, 1/3),\ \begin{bmatrix} -\lambda_1 & 0 \\ 0 & -\lambda_3 \end{bmatrix}\right\} ,$$

$$(\underline{\alpha}_3, T_3) = \left\{(1/3, 2/3),\ \begin{bmatrix} -\lambda_2 & 0 \\ 0 & -\lambda_3 \end{bmatrix}\right\} ,$$

$$(\underline{\alpha}_4, T_4) = \left\{ (1/3,1/3,1/3), \begin{bmatrix} -\lambda_1 & 0 & 0 \\ 0 & -\lambda_2 & 0 \\ 0 & 0 & -\lambda_3 \end{bmatrix} \right\} ,$$

then we may either choose

$$(\underline{\zeta}_j, Z_j) = (\underline{\alpha}_j, T_j) , \qquad j = 1, 2, 3,$$

with

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2/5 & 1/5 & 2/5 \end{bmatrix} ,$$

or alternatively

$$(\underline{\zeta}_j, Z_j) = \{(1), (-\lambda_j)\} , \qquad j = 1, 2, 3,$$

in which case D is given by

$$D = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 2/3 & 0 & 1/3 \\ 0 & 1/3 & 2/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix} .$$

Lemma 4

The covariance between X_n and X_{n+m}, for $n \geq 0$, $m \geq 1$, is given by

$$\text{Cov}\ (X_n, X_{n+m}) = \underline{\mu}^{(1)}\ \Delta(\underline{\gamma})\ (P^m - \Gamma)\ \underline{\mu}^{(1)} , \tag{12}$$

and also by

$$\text{Cov}\ (X_n, X_{n+m}) = \underline{\mu}^{*(1)}\ D^T\ \Delta(\underline{\gamma})\ (P^m - \Gamma)\ D\ \underline{\mu}^{*(1)} , \tag{13}$$

where for any vector $\underline{x}$, the matrix $\Delta(\underline{x})$ is defined by diag $(x_1,\ldots,x_N)$.

Proof. By Lemma 3, the covariance is equal to

$$\begin{aligned}\text{Cov}\,(X_n,X_{n+m}) &= E(X_n,X_{n+m}) - E(X_n)\,E(X_{n+m}) \\ &= \sum_{i=1}^{N}\sum_{j=1}^{H} (\gamma_i (P^m)_{ij} \mu_i^{(1)} \mu_j^{(1)} - \gamma_i\gamma_j \mu_i^{(1)} \mu_j^{(1)}) ,\end{aligned}$$

which proves (12). Equation (13) results from (11).

From Lemmas 3 and 4, we conclude that, for the process $\tilde{P}$, the intervals of time between events are identically distributed and are in general correlated. If the Markov chain P is aperiodic, then $\lim_{m\to\infty} P^m = \Gamma$, and the covariance of X_n and X_{n+m} tends to zero as m tends to infinity.

To conclude this section, we shall now examine under what condition the process $\tilde{P}$ is a renewal process. If $\tilde{P}$ is a renewal process, then it is characterised by its interrenewal distribution $r(\cdot)$. The distribution $r(\cdot)$ *is PH and is represented as* $r(x) = \underline{\gamma}\,\underline{F}(x)$, for $x \geq 0$, (by Lemma 3) or as $r(x) = \underline{\gamma}\,D\,\underline{G}(x)$, for $x \geq 0$, where $(G_1(x),\ldots,G_L(x))$ is any set of linearly independent PH-distributions satisfying (10), and D is the corresponding matrix. The next theorem holds for any choice of $\underline{G}(x)$.

Theorem 1

The process $\tilde{P}$ is a renewal process if and only if the following property holds.

For all $k \geq 2$, and for all $(\tau_1,\ldots,\tau_k)$ satisfying $1 \leq \tau_i \leq L$, for $i = 1, \ldots, k$,

$$\underline{\gamma}\;\Delta_{\tau_1}\prod_{i=2}^{k}[(P-\Gamma)\Delta_{\tau_i}]\,\underline{e} = 0 , \tag{14}$$

or equivalently,

$$\underline{\gamma}\;\Delta_{\tau_1}\Big(\prod_{i=2}^{k} P\;\Delta_{\tau_i}\Big)\,\underline{e} = \prod_{i=1}^{k}(\underline{\gamma}\,\underline{D}_{\cdot\tau_i}) , \tag{15}$$

where $\underline{D}_{\cdot j}$ represents the jth column of the matrix D, and $\Delta_j = \text{diag}(\underline{D}_{\cdot j})$.

Proof. The proof is very simple. Since the variables $\{X_n, n \geq 0\}$ are identically distributed, $\tilde{P}$ is a renewal process if and only if for all $k \geq 2$, the random variables $X_1, X_2, \ldots, X_k$ are independent, which is true if and only if

$$P[\bigcap_{i=1}^{k} \{X_i \leq x_i\}] = \prod_{i=0}^{k} P[X_i \leq x_i], \qquad \text{for all } x_1, \ldots, x_k \geq 0 ,$$

equivalently, if

$$\sum_{\substack{\nu_i=1 \\ 1\leq i\leq k}}^{N} \{\gamma_{\nu_1} F_{\nu_1}(x_1) \prod_{j=2}^{k} P_{\nu_{j-1}\nu_j} F_{\nu_j}(x_j) - \prod_{j=1}^{k} \gamma_{\nu_j} F_{\nu_j}(x_j)\} = 0 ,$$

for all $x_1, \ldots, x_k \geq 0$, if and only if (by (10))

$$\sum_{\substack{\nu_i=1 \\ 1\leq i\leq k}}^{N} \sum_{\tau_i=1}^{L} \{\gamma_{\nu_1} d_{\nu_1\tau_1} [\prod_{j=2}^{k} (P_{\nu_{j-1}\nu_j} - \gamma_{\nu_j}) d_{\nu_j\tau_j}] \cdot \prod_{j=1}^{k} G_{\tau_j}(x_j)\} = 0 , \quad \text{for all } x_1, \ldots, x_k \geq 0 .$$

Since the PH-distributions $G_i(\cdot)$ are linearly independent, this holds if and only if

$$\sum_{\substack{\nu_i=1 \\ 1\leq i\leq k}}^{N} \{\gamma_{\nu_1} d_{\nu_1\tau_1} [\prod_{j=2}^{k} (P_{\nu_{j-1}\nu_j} - \gamma_{\nu_j}) d_{\nu_j\nu_j}]\} = 0 ,$$

for all $\tau_1, \ldots, \tau_k$ such that $1 \leq \tau_i \leq L$. The condition (14) is now obvious, and it is a simple matter to prove that (14) and (15) are equivalent.

This theorem provides us with a technical condition which is not very attractive. The following corollary is more interesting and useful for modeling purposes.

Corollary 1

For $\tilde{P}$ to be a renewal process, it is sufficient that

$$(P-\Gamma)\, D = 0 \,, \tag{16}$$

or that

$$D^T \Delta(\underline{\gamma})\, (P-\Gamma) = 0 \,. \tag{17}$$

If L is equal to one, then both conditions are always satisfied and $\tilde{P}$ is always a renewal process.

If L is equal to N, then both conditions are necessary, as $\tilde{P}$ is a renewal process if and only if the matrix $P-\Gamma$ is equal to zero.

Proof. The condition (16) is obviously sufficient, as $(P-\Gamma)\underline{D}_{\cdot i} = \underline{0}$ for all i implies that $(P^n-\Gamma)\, \Delta_i\, \underline{e} = P^{n-1}(P-\Gamma)\, \underline{D}_{\cdot i} = \underline{0}$ for each $n \geq 1$ and each $i = 1, \ldots, L$, which in turn implies (14).

Similarly, the condition (17) is sufficient, since

$$\underline{\gamma}\, \Delta_i (P^n-\Gamma) = \underline{\gamma}\, \Delta_i (P-\Gamma) P^{n-1} = \underline{D}_{\cdot i}^T\, \Delta(\underline{\gamma})\, (P-\Gamma) P^{n-1} \,.$$

If L is equal to one, then $D = \underline{e}$ and both conditions (16) and (17) are satisfied.

If L is equal to N, then D may be chosen equal to I, and both (16) and (17) reduce to $P-\Gamma = 0$. The necessary part of the condition results from (14): if $k = 1$, then $\underline{\gamma}\, \Delta_i (P-\Gamma)\, \Delta_j \underline{e}$ must be equal to zero for all $i, j = 1, \ldots, N$. As $D = I$, this reduces to $\gamma_i (P_{ij} - \gamma_j) = 0$, for all $i, j = 1, \ldots, N$. Since $\gamma_i > 0$ for all i, the Corollary is proved.

Remarks

1. One easily proves that condition (16) holds if and only if there exist a vector $\underline{v}$ such that $P D = \underline{e} \cdot \underline{v}$. Therefore, it is not necessary to determine $\underline{\gamma}$ in order to check whether (16) holds or not.

2. The conditions (16) and (17) are not equivalent. Simple examples exist, for which one of the conditions holds, but not the other.

3. If neither (16) nor (17) holds, the combined condition,

$$D^T \Delta(\underline{\gamma}) (P-\Gamma) D = 0 , \tag{18}$$

is not a sufficient condition for $\tilde{P}$ to be a renewal process (this is proved by a counterexample), but is the necessary and sufficient condition for two successive intervals of time to be independent.

Simon [7] examines equivalences for Markov-renewal processes and, in particular, the conditions under which a Markov-renewal process is equivalent to a renewal process. It appears that for our process $\tilde{P}$, Theorems 2.2.1, 2.2.2 and 2.2.9 [7] respectively correspond to the sufficient condition (17), and to the cases $L = 1$ and $L = N$ in Corollary 1. From Theorem 2.2.8 [7], results the following necessary condition. There exist a non-negative vector $\underline{w}$ such that

$$D^T(\Delta(\underline{w}) P - \underline{\gamma}^T \cdot \underline{w}) = 0 ,$$

moreover, if condition (17) holds, that vector clearly is $\underline{\gamma}$; if condition (16) holds, we do not know $\underline{w}$, but the right eigenvector corresponding to $r(t)$ (in the terminology of [7]), is $\underline{F}(t)$.

Because of the special structure of our point process, and in particular because the distributions, we consider, are PH, we have obtained conditions which do not have to be examined for each t.

Finally, let us mention that Simon [7] introduces and analyses the notion of collapsibility, which generalizes to Markov renewal processes the notion of lumpability for Markov chains - see Kemeny and Snell [2]. If the matrix D contains only entries equal to zero or one, this corresponds to the notion of collapsibility. As we have observed, the entries of D may take any real values. If those values were all positive, one might consider that D corresponds to some sort of randomized collapsibility, since the row sums of D are equal to one. This correspondence appears difficult to extend to the case where D contains negative entries.

4. Examples

4.1 Exponential distributions

The case where the PH-distributions $(\underline{\alpha}_i, T_i)$ are exponential, respectively with parameter λ_i, is particularly simple. We then immediately obtain that

$$\overline{T} = -\Delta(\underline{\lambda}) \ , \quad \overline{A} = P \ , \quad m^* = (\underline{\gamma}\, \underline{\lambda}^{-1})^{-1} \ ,$$

$$\pi_i = m^* \gamma_i \lambda_i^{-1} \ , \quad \text{for} \quad i = 1, \ldots, N \ ,$$

where the vectors $\underline{\lambda}$ and $\underline{\lambda}^{-1}$ respectively have components equal to λ_i and λ_i^{-1}. After some simple manipulations, (9) reduces to

$$\begin{aligned} m(t) = {} & m^* t\, \underline{e} + m^* (I - m^* \Gamma\, \Delta(\underline{\lambda}^{-1}))\, (I - P + \Gamma)^{-1}\, \underline{\lambda}^{-1} \\ & + m^* [m^* \Gamma\, \Delta(\underline{\lambda}^{-1}) - \exp(-\Delta(\underline{\lambda})\, (I-P)t)]\, (I - P + \Gamma)^{-1}\, \underline{\lambda}^{-1} \ . \end{aligned} \tag{19}$$

Exponential distributions with different parameters are linearly independent. Therefore, the distributions $\{(\underline{\zeta}_j, Z_j),\ 1 \le j \le L\}$ may be chosen to be the set of different exponential distributions in $\{(\underline{\alpha}_i, T_i),\ i = 1, \ldots, N\}$, and the matrix D has a very simple structure: each element of D is either

equal to zero or one, each row of D contains exactly one element equal to one, each column of D contains at least one element equal to one. We may then strengthen Corollary 1.

Corollary 1'.

If the entries of D are each equal to zero or one, and if $L = N - 1$, then a necessary and sufficient condition for $\tilde{P}$ to be a renewal process is that one at least of Equations (16) or (17) holds.

Proof. The technical proof is belabored. We do reproduce it here since it cannot be extended to $L \leq N - 2$.

If the matrix P has identical rows, then $\tilde{P}$ is obviously a renewal process, with hyperexponential intervals between events, and the matrix $(I-P+\Gamma)^{-1}$ in (9) may be replaced by the identity matrix.

4.2 Platooned events

Let us assume that the process consists of groups of events, the number of events in a group has a discrete PH-distribution $(\underline{f},F)$, the intervals of time between events in a given group have PH-distribution $(\underline{\alpha},A)$, while the intervals of times between groups have a PH-distribution $(\underline{\beta},B)$. Such a process may be used to model platooned arrivals to a system, as is done in Neuts and Chakravarthy [6]. Then,

$$P = \left(\begin{array}{c|c} F & \underline{e}-F\underline{e} \\ \hline \underline{f}F & 1-\underline{f}F\underline{e} \end{array}\right) , \qquad (20)$$

P is a square matrix of order N if F is a matrix of order $N - 1$; $(\underline{\alpha}_i, T_i) = (\underline{\alpha},A)$, for $i = 1, \ldots, N - 1$, and $(\underline{\alpha}_N, T_N) = (\underline{\beta},B)$.

Simple calculations yield that the stationary probability vector $\underline{\gamma}$ of P is given by $\underline{\gamma} = (\tilde{\underline{\gamma}}, \gamma_N)$, where

$$\tilde{\underline{\gamma}} = \underline{v}\, F\ , \quad \gamma_N = 1 - \tilde{\underline{\gamma}}\, \underline{e}\ , \tag{21}$$

$$\underline{v} = (\underline{f}(I-F)^{-1}\underline{e})^{-1}\, \underline{f}(I-F)^{-1}\ , \tag{22}$$

and that

$$m^* = [(\tilde{\underline{\gamma}}\, \underline{e})\ (-\underline{\alpha}\, A^{-1}\, \underline{e}) + \gamma_N\ (-\underline{\beta}\, B^{-1}\, \underline{e})]^{-1}\ ,$$

$$\underline{\pi}_i = -m^*\, \tilde{\gamma}_i\, \underline{\alpha}\, A^{-1}\ , \qquad \text{for } i = 1, \ldots, N-1\ ,$$

$$= -m^*\, \gamma_N\, \underline{\beta}\, B^{-1}\ , \qquad \text{for } i = N\ .$$

The expression (9) does not simplify much.

If the PH-distributions $(\underline{\alpha},A)$ and $(\underline{\beta},B)$ are different, then we clearly may choose D equal to

$$D = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ \vdots & \vdots \\ 1 & 0 \\ 0 & 1 \end{pmatrix}\ , \tag{23}$$

and we may complete the corollary 1 as follows.

Corollary 1"

If the matrix D is given by (23), then the process $\tilde{P}$ is a renewal process if and only if

$$^{T}D\ \Delta(\underline{\gamma})\ (P^n - \Gamma)\ D = 0\ , \qquad \text{for all } n \geq 1\ . \tag{24}$$

Proof. We partition the matrix P and the vector $\underline{\gamma}$ as

$$P = \left(\begin{array}{c|c} P_{11} & \underline{P}_{12} \\ \hline \underline{P}_{21} & P_{22} \end{array}\right)\ , \qquad \underline{\gamma} = (\underline{\gamma}_1, \gamma_2)\ .$$

Simple calculations yield that

$$\underline{\gamma}_1 = \gamma_2 \, \underline{P}_{21} \, (I-P_{11})^{-1} \, , \quad \gamma_2 = (1 + \underline{P}_{21} \, (I-P_{11})^{-1} \, \underline{e})^{-1} \, .$$

a. (24) is a necessary condition. This is easily proved as follows. For $n = 1$, it is equivalent to (18), the condition for two successive intervals to be independent. For $n = 2$, we consider (14) with $k = 3$. Since $\underline{\gamma} \, \Delta_{\tau} \, (P-\Gamma) \, \Delta_1 \, (P-\Gamma) \, \Delta_{\tau_3} \, \underline{e} + \underline{\gamma} \, \Delta_{\tau_1} \, (P-\Gamma) \, \Delta_2 \, (P-\Gamma) \, \Delta_{\tau_3} \, \underline{e} = \underline{\gamma} \, \Delta_{\tau_1} \, (P-\Gamma)^2 \, \Delta_{\tau_3} \, \underline{e}$ must be equal to zero, this proves (24) for $n = 2$. Similarly, we prove that (24) is necessary for larger values of n.

b. (24) is a sufficient condition. It is easy, although tiresome, to show that (24) implies that

$$\underline{\gamma}_1 \, C_1^{n+1} \underline{e} = (\underline{P}_{21}-\underline{\gamma}_1) \, C_1^n \, \underline{e} = 0 \, , \qquad \text{for all} \quad n \geq 0 \, , \tag{25}$$

where $C_1 = P_{11} - \underline{e} \cdot \underline{\gamma}_1$. This in turn implies that

$$(P-\Gamma)^n = \left(\begin{array}{c|c} C_1^n - \sum_{\nu=1}^{n-1} C_1^{\nu} \underline{e} \cdot (\underline{P}_{21}-\underline{\gamma}_1) \, C_1^{m-1-\nu} & -C_1^n \, \underline{e} \\ \hline (\underline{P}_{21}-\underline{\gamma}_1) \, C_1^{n-1} & 0 \end{array} \right) \, .$$

It is then a simple matter to prove that for ν equal either to 1 or 2,

$$(P-\Gamma)^n \, \Delta_2 \, (P-\Gamma)^m \, \Delta_{\nu} \, \underline{e} = 0 \, ,$$

and $(P-\Gamma)^n \, \Delta_1 \, (P-\Gamma)^m \, \Delta_{\nu} \, \underline{e} = (P-\Gamma)^{n+m} \, \Delta_{\nu} \, \underline{e}$, for $n,m \geq 1$. Therefore, the left-hand side of (14) is equal to zero if $\tau_i = 2$ for any $i = 2, \ldots, k-1$ and is otherwise equal to

$$({}^{T}D \, \Delta(\underline{\gamma}) \, (P-\Gamma)^{k-1} \, D)_{\tau_1,\tau_k} \, ,$$

This completes the proof.

Remark. This corollary has the following simple interpretation. If D is given by (23), then $N - 1$ of the distributions $\{(\underline{\alpha}_i, T_i), i = 1, \ldots, N\}$ have a "common" type, the last one has an "odd" type. Let N_1 denote the number of intervals of the common type between two consecutive intervals of the odd type, and let N_2 similarly denote the number of intervals of the odd type between two intervals of the common type. From Corollary 1", it results that

> the process $\tilde{P}$ is a renewal process if and only if N_1 and N_2 both have a geometric distribution, the parameters being respectively equal to $(1-P_{22})$ and P_{22}.

Clearly, N_2 has a geometric distribution with parameter P_{22}. The equations (25) may be written as

$$\underline{P}_{21} P_{11}^n \underline{e} = (\underline{\gamma}_1 \underline{e})^{n+1} , \qquad \text{for } n \geq 0 . \tag{26}$$

As $P[N_1 = n] = \underline{P}_{21} (P_{11}^{n-1} - P_{11}^n) \underline{e} = (\underline{\gamma}_1 \underline{e})^n (1 - \underline{\gamma}_1 \underline{e}) = (1-\gamma_2)^n \gamma_2 = (1-P_{22})^n P_{22}$, this completes this proof.

In the present case, from the equations (20-22) and (26), it results that the process of platooned events is a renewal process if and only if

$$\underline{f} F^n \underline{e} = (1 - (\underline{f}(I-F)^{-1} \underline{e})^{-1})^n , \qquad \text{for } n \geq 0 ,$$

where $1 + \underline{f} (I-F)^{-1} \underline{e}$ is the expected number of events in a platoon.

4.3 The Interrupted Poisson Process

This process is used in models for telephone engineering (Heffes [1]). We consider a process in a random environment, with two alternating environment states. Both states have exponential duration, with parameters σ_1 and σ_2 respectively. While the process is in the first environment state (on-state),

an independent Poisson process of rate λ is turned on. When the environment is in the second state (off-state), no arrivals can occur. Neuts and Chakravarthy [6] have shown that the interrupted Poisson process is a special case of the platooned events process, and that the number of events in a platoon is geometric, therefore, the stationary interrupted Poisson process is a renewal process. This was already shown by Kuczura [3] by different methods.

In fact, it appears that the interrupted Poisson process is a very special case of the type of processes analysed in the present paper. From [6], one observes that the process can be described by two states and two PH-distributions, the matrix P and the PH-distributions are as follows.

$$P = \begin{vmatrix} \frac{\lambda}{\lambda+\sigma_1} & \frac{\sigma_1}{\lambda+\sigma_1} \\ \\ \frac{\lambda}{\lambda+\sigma_1} & \frac{\sigma_1}{\lambda+\sigma_1} \end{vmatrix} , \tag{27}$$

$$\underline{\alpha}_1 = (1) , \quad T_1 = (-\lambda - \sigma_1) ,$$

$$\underline{\alpha}_2 = (1,0,0) , \quad T_2 = \begin{vmatrix} -\lambda-\sigma_1 & \lambda+\sigma_1 & 0 \\ 0 & -\lambda-\sigma_1 & \lambda+\sigma_1 \\ 0 & \sigma_1\sigma_2(\lambda+\sigma_1)^{-1} & -\sigma_2 \end{vmatrix} .$$

The first state of the Markov chain P corresponds to the following event {the interrupted Poisson process is in the on-state, and an arrival will occur before the end of the on-state}. The second state corresponds to the following composite event {the interrupted Poisson process is in the on-state, no arrivals will occur before the next off-state or the process is in the off-state, or the process has returned to the on-state and an arrival will occur before the next off-state}. From the structure (27) of P, it is

obvious that the interrupted Poisson process is a renewal process.

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A semi-Markovian point process is defined, for which the intervals of time between successive events have phase type distribution. The distribution of the number of events in an interval is examined, and it is shown how the expected number of events in an interval may be efficiently computed. A stationary version of the process is analyzed. In particular, the necessary and sufficient condition under which the new process is a renewal process is determined. Simple sufficient conditions are presented.

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